CONTINUOUS VERSUS DISCRETE QUASI-ENERGY SPECTRUM IN THE QUANTAL DESCRIPTION OF A SIMPLE PARAMETRIC RESONATOR

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We investigate the quasi-energy spectrum of a quantal parametric resonator, whose frequency is modulated by a periodic $\delta(t)$ term. The system displays two distinct phases, depending on the nature (discrete or continuous) of the quasi-energy spectrum. We investigate the dynamics in the two phases and the transition between them which is induced by varying the coupling strength. Special attention is given to those observables which might be used as indicators of stochasticity in the quantum dynamics.

The quantum mechanical (QM) description of classically irregular (stochastic) systems became recently a subject of intensive research [1]. Much attention is directed towards the discussion of simple (one-dimensional) systems which are perturbed by an external, time dependent and periodic force [2]. For such systems, some detailed comparisons between the QM and the classical descriptions are available, but till now it is not entirely clear which, and to what extent, classical chaotic features survive in the QM description [3]. The QM propagator for periodically perturbed systems can always be written as [4]

$$K(t) = P(t) \exp(-\mathrm{i}Gt/\hbar) , \tag{1}$$

where $P(t)$ is a unitary and periodic operator, while G is a time independent hermitian operator. G is the so-called quasi-energy (q.e.) operator. The development of the system in time is entirely determined by the spectral properties of the q.e. operator. As long as the q.e. spectrum is discrete (pure-point), the wave function is quasi-periodic [5], and the phase-space diffusion which characterized the corresponding classical description is limited in the QM treatment to the early, transient stages of the motion [6]. Much less is known about the role played by the continuous part of the q.e. spectrum in determining the quantum dynamics. Continuous bands in the q.e. spectrum are identified in, e.g., the case of the periodically kicked rotor, when the kick frequency and the fundamental rotation frequency are rotationally related [7]. The condition for the appearance of the continuous bands depends exclusively on the time structure of the system (resonance condition), and not on the strength or shape of the perturbing force.

In the present note we would like to present a simple system in which the q.e. spectrum makes a transition from a discrete (pure-point) phase to an absolute continuous phase, as the coupling strength is continuously varied across a critical value. The transition between the two regimes and its implications for the dynamics of the system can be analytically discussed and compared with the classical description.

We consider a harmonic oscillator with

$$H_0 = p^2/2m + \tfrac{1}{2}m\omega^2 x^2 ,$$

whose frequency is modulated by [‡1]

[‡1] A similar problem was discussed by Berry et al. [2], where the discussion was limited to the discrete q.e. spectrum.

$$V(x, t) = \tfrac{1}{2} m\omega\beta x^2 \sum_n \delta(t - nT) .$$

The one-period propagator and the corresponding q.e. operator are defined through

$$\exp(-\mathrm{i}GT/\hbar) = \exp[(-\mathrm{i}/\hbar)\tfrac{1}{2}m\omega\beta x^2] \exp[(-\mathrm{i}/\hbar)H_0 T] . \tag{2}$$

The operators x^2, H_0 and $(xp + px)$ form a Lie group [8] (isomorphic to SU(1,1)), and therefore G can be expressed as a linear combination of the group generators

$$G = \lambda H_0 + \mu x^2 + \nu(xp + px) , \tag{3}$$

with λ, μ, ν as yet undetermined. G is quadratic in p and x and therefore its spectral properties are well understood. If G is a definite quadratic form, then its spectrum is that of a harmonic oscillator. It is discrete and the eigen-functions are expressed in terms of Hermite polynomials. If G is a non-definite quadratic form then its spectrum is that of an inverted harmonic oscillator. It is absolutely continuous and the eigen-functions are expressed in terms of parabolic cylindrical functions. In either case, the explicit form of the eigen-functions depends on the parameters λ, μ, ν. To get the relation between λ, μ, ν and the physical parameters T, ω and β, we take matrix elements of both sides of eq. (2), with G as in eq. (3). The integrals can be analytically performed [9] and the following expression for G emerges,

$$G = s[p^2/2m + \tfrac{1}{2}m\omega^2(1 + \beta \cot \omega T)x^2 + \tfrac{1}{4}\omega\beta(xp + px)] , \tag{4}$$

where

$$s = \arcsin(r \sin \omega T)/r\omega T , \quad r^2 = 1 + \beta \cot \omega T - \tfrac{1}{4}\beta^2 . \tag{5}$$

r^2 can take either positive or negative values. For $r^2 < 0$, r is purely imaginary and the above expressions are analytically continued to the imaginary r axis. The sign of r^2 determines the signature of G and hence its spectral properties. This can become apparent by applying a local gauge transformation

$$W(x) = \exp[-\tfrac{1}{4}\mathrm{i}\beta(x/x_0)^2] , \quad x_0 = (\hbar/\omega m)^{1/2} , \tag{6}$$

which eliminates the mixed term in G:

$$G = sW(x)(p^2/2m + \tfrac{1}{2}m\omega^2 r^2 x^2)W^+(x) . \tag{7}$$

Thus, when $r^2 > 0$ the q.e. spectrum is discrete, while for $r^2 \leqslant 0$ it is continuous. The factor s (which is always real), scales the q.e. spectrum and it may vanish only at resonance, $\omega T = \pi n$. We shall exclude the resonance case from the discussion. For any value of ωT $(\neq \pi n)$ we can define two critical coupling strengths $\beta^*_\pm = 2(\cos \omega T \pm 1)/\sin \omega T$ where $r^2(\beta^*) = 0$. When β is varied, the system undergoes a "phase transition" when β crosses the values β^*_+ or β^*_-. This transition will be discussed below.

The N step propagator can be explicitly written in the coordinate representation

$$K(x, x'; N) \equiv \langle x| \exp[-(\mathrm{i}/\hbar)TNG] |x'\rangle = W(x)\langle x | \exp[-(\mathrm{i}/\hbar)(p^2/2m + \tfrac{1}{2}m\omega^2 r^2 x^2)sTN] |x'\rangle W^+(x')$$

$$= W(x) [r/2\mathrm{i}\pi x_0^2 \sin \eta]^{1/2} \exp\{(r/2\mathrm{i}x_0^2 \sin \eta)[(x^2 + x'^2) \cos \eta - 2xx']\} W^+(x') , \tag{8}$$

where

$$\eta = N/N_0 , \quad N_0 = (\omega T s r)^{-1} .$$

Given any initial wave function $\Psi_0(x)$, the wave function $\Psi_N(x)$ after N kicks will be

$$\Psi_N(x) = \int \mathrm{d}x' \, K(x, x'; N)\Psi_0(x') .$$

Taking $\Psi_0(x)$ as the ground state of H_0 we get

$$\Psi_N(x) = (-v/\pi x_0^2)^{1/4} \exp\{\tfrac{1}{2}\mathrm{i} \arctan a - \tfrac{1}{2}v(x/x_0)^2 + \tfrac{1}{2}\mathrm{i}[a(1-v)-\beta](x/x_0)^2\}\,, \tag{9}$$

where

$$a = r\cot\eta + \tfrac{1}{2}\beta\,, \quad v = r^2[\sin^2\eta + (r\cos\eta + \tfrac{1}{2}\beta\sin\eta)^2]^{-1}\,.$$

The following properties can be derived.

(a) *Correlation functions*. The overlap $\langle\Psi_N|\Psi_0\rangle$ depends on the time N through the parameters a and v

$$|\langle\Psi_N|\Psi_0\rangle|^2 = \left(\frac{4v}{(1+v)^2 + [a(1-v)-\beta]^2}\right)^{1/4}\,. \tag{10}$$

For $r^2 > 0$ η is real and v is a positive, finite and periodic function of N. Thus the correlation function is a periodic function of N whose period is of the order $N_0/2\pi$. For $r^2 < 0$ η is purely imaginary and v decays to zero exponentially. The correlation function also decays with a time scale which is again characterized by $|N_0|$. Approaching the critical coupling strength β^* from below or above we find that the memory time (or the period) are singular at β^*, with $|N_0| \sim |\beta - \beta^*|^{-1/2}$. We can therefore consider the system to undergo a transition from the "periodic phase" to the "unbounded phase", and the behaviour of the correlation time $|N_0|$ near the critical point is reminiscent of a second order phase transition.

(b) *Energy transfer*. The first and the second moments of the energy distribution read

$$\langle\Psi_N|H_0|\Psi_N\rangle = (\hbar\omega/4v)(1+v^2+d^2)\,, \quad \langle\Psi_N|H_0^2|\Psi_N\rangle = (\tfrac{1}{2}\hbar\omega)^2[1+3d^2+(3/4v^2)(v^2-d^2-1)^2]\,,$$
$$d = a(1-v)-\beta\,. \tag{11}$$

In the periodic phase ($r^2 > 0$) the expectation value of H_0 remains bounded and the variance of the energy distribution oscillates about a finite mean. In the unbounded phase ($r^2 < 0$), the mean energy and the variance increase exponentially as $N \to \infty$, but

$$(\langle\Psi_N|H_0^2|\Psi_N\rangle - \langle\Psi_N|H_0|\Psi_N\rangle^2)^{1/2} \to \sqrt{2}\langle\Psi_N|H_0|\Psi_N\rangle\,. \tag{12}$$

(c) *Probability distribution*. States with odd occupation number are not excited (x^2 is an even operator). The probability to find the system in the state $|2n\rangle$ of H_0 after the Nth kick is given by

$$P_n(N) = [(2n)!/(2^n n!)^2](1-\bar{v})^n\bar{v}^{1/2} \approx (\pi n)^{-1/2}\bar{v}^{1/2}\exp(-n\bar{v})\,, \quad \bar{v} = 4v\{(1+v)^2 + [a(1-v)-\beta]^2\}^{-1}\,.$$

Because of the periodic character of v for $r^2 > 0$, $P_n(N)$ is periodic in N and the excitation probability is concentrated in a finite number of states. For $r^2 < 0$, v decays exponentially and the probability distribution spreads over a larger and larger number of states. in the interval $n \leqslant \bar{v}^{-1} \approx \exp(N/N_0)$, the relative probability to populate the states approaches an "equilibrium" situation, for which

$$P_n(N) = \frac{W_{n,n+1}P_{n+1}(N) + W_{n,n-1}(N)}{W_{n,n+1} + W_{n,n-1}}\,.$$

Here

$$W_{n,n\pm1} = |\tfrac{1}{2}m\omega\beta\langle 2n|x^2|2(n\pm1)\rangle|^2/\hbar^2$$

is the rate per kick for the transition $|2n\rangle \to |2(n\pm1)\rangle$. Such behaviour is typical for systems obeying stochastic rate-equations when they approach their steady state (equilibrium).

(d) *Wigner transform*. The QM analogue of the phase space distribution function is the Wigner transform

$$W_N(p,x) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \exp[-(\mathrm{i}/\hbar)ps]\,\Psi_N(x+\tfrac{1}{2}s)\Psi_N^*(x-\tfrac{1}{2}s)\,\mathrm{d}s$$
$$= (1/\pi\hbar)\exp[-v(x/x_0)^2]\,\exp\{\{[a(1-v)-\beta]x/x_0 - px_0/\hbar\}^2/v\}\,. \tag{13}$$

The Wigner transform for $r^2 > 0$ is localized in a bounded area of phase-space. In the unbounded phase ($r^2 < 0$) and for large N, the Wigner transform is concentrated along the line $p = m\omega(|r| - \frac{1}{2}\beta)x$, for $x < x_0 \exp(N/N_0)$. Thus, phase-space is not covered homogeneously, but rather along a thin and long strip, indicating a very strong phase-space correlation.

The properties (a)–(d) show that the QM treatment clearly distinguishes between the two phases of the system, which are different in their q.e. spectra as well as in all the other properties investigated. The unbounded phase, although displaying some properties which might be attributed to a stochastic mechanism (exponential decay of correlation function and the "equilibration" of the probability distribution), does not correspond to our intuitive understanding of chaotic systems. This is best manifested in the strong correlations in the Wigner transform.

The classical description of the same system is governed by equations of motion, which, in the present case take the form of a discrete mapping

$$x_{n+1} = \cos \omega T x_n + (\sin \omega T/m\omega) p_n , \quad p_{n+1} = -m\omega(\sin \omega T + \beta \cos \omega T)x_n + (\cos \omega T - \beta \sin \omega T)p_n . \tag{14}$$

This is a linear area preserving mapping. The eigenvalues of the stability matrix are unimodular for $r^2 > 0$. For $r^2 < 0$ one eigenvalue always exceeds unity in absolute magnitude, bringing about a positive Lyapunov exponent for all trajectories. This fact does not imply here that the motion is irregular. Indeed, one can easily check that the quadratic form

$$I(x_n, p_n) = p_n^2/2m + \tfrac{1}{2}m\omega^2(1 + \beta \cot \omega T)x_n^2 + \tfrac{1}{4}\omega\beta(x_n p_n + p_n x_n) \tag{15}$$

is an integral of the motion ‡2. Comparing eq. (15) with eq. (4) we find that up to the scale factor s, $I(x_n, p_n)$ is the quasi-energy. The classical motion remains bounded when it is definite, and diverges on a hyperbola in phase-space if the quasi-energy is indefinite. (Asymptotically, $p_n = m\omega(|r| - \frac{1}{2}\beta)x_n$ which is identically the same relation obtained from the QM Wigner function. The QM and the classical dynamics match so well in this system because of the quadratic dependence of the hamiltonian on x and p [10]. The existence of a classical constant of the motion proves the integrability of the classical problem, and rules out the possibility of classical irregularity.

The moral we can draw from the above discussion is that like in the resonance situations in the QM description of the kicked rotor, the continuous component of the q.e. spectrum brings about the features which are inherent to the fact that the system is not closed. There is no evidence in these systems for a stochastic QM behaviour which is due to the appearance of a continuous component in the q.e. spectrum.

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‡2 We are indebted to Dr. S. Fishman for pointing this out to us.

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