BOUND STATES OF NEUTRONS IN THE CYLINDRICALLY SYMMETRIC MAGNETIC FIELD OF A THIN CURRENT CARRYING WIRE

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We prove that neutral particles with a magnetic dipole moment possess both an infinite number of bound states and a continuum of unbound states in the magnetic field of a current carrying wire (quantum trap for neutral particles) The (magnetic) transitions between the bound states can be used to determine the dipole moment of the particles by rf absorption spectroscopy. The magnetically bound states forming on the surface of the wire resemble Rydberg states in extension and shape Applications to quantum chaos research are discussed.

More than twenty years ago a curious system was proposed: electrons trapped at the surface of liquids or solids by their image charge [1]. Due to the translational invariance parallel to the surface, the binding is effective only in the direction orthogonal to the surface and the "surface state electrons" (SSE) form two-dimensional sheets hovering above the surface. In a sequence of beautiful experiments sheets of SSE were found to exist on the surface of liquid ^{4}He [2] as well as on metal surfaces [3,4]. The binding energies of the SSE could be measured to high precision by means of rf absorption [2] and photoemission [3,4] spectroscopy. Recently, even Wigner crystallization [5] of the two-dimensional electron sheets was demonstrated experimentally [6].

In this article we show that the existence of surface states is not only restricted to the domain of charged particles. We demonstrate that even neutral particles can form bound states hovering above a macroscopic surface, provided they possess a magnetic dipole moment. To this end we propose the magnetic analog of the SSE system: neutral particles bound to the surface of a current carrying wire by the interaction of their magnetic dipole moment with the wire's circular magnetic field. Classically, these bound states correspond to a dipole periodically bouncing off the surface of the wire. Quantizing this motion results in magnetically bound states (MBS). Like in the SSE case, the MBS spectrum is hydrogenic and in the case of neutrons, e.g., and for realistic currents, the binding energies are in the radio frequency region. The MBS system might be useful to determine the magnetic moments of neutral particles by rf absorption spectroscopy. A more exotic application would be the study of the neutral fermion analog of a Wigner crystal [5] where only spin–spin interactions are present.

The magnetic field for the MBS system is generated by a thin wire, infinitely extended in the z direction which also serves as the quantization axis.

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Due to the cylindrical symmetry of the problem, we introduce cylindrical coordinates:

$$x=\rho\cos\phi,\quad y=\rho\sin\phi,\quad z\equiv z. \tag{1}$$

In these coordinates (and in SI units) the magnetic field of the wire is given by [7]

$$\boldsymbol{B}=\frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{e}}_\phi, \tag{2}$$

where μ_0 is the magnetic permeability of the vacuum, I is the current in the wire and $\hat{\boldsymbol{e}}_\phi$ is the unit vector in ϕ direction, explicitly given by

$$\hat{\boldsymbol{e}}_\phi=\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}. \tag{3}$$

The interaction energy of a magnetic dipole $\boldsymbol{\mu}$ with the magnetic field $\boldsymbol{B}$ is given by

$$V=-\boldsymbol{\mu}\cdot\boldsymbol{B}. \tag{4}$$

For a spin-$\frac{1}{2}$ particle of mass M we have

$$\boldsymbol{\mu}=g\mu_M\boldsymbol{s},\quad \boldsymbol{s}=\tfrac{1}{2}\boldsymbol{\sigma}, \tag{5}$$

where $\hat{\sigma}_j$ are the Pauli matrices, and $g/2$ is the magnetic moment of a spin-$\frac{1}{2}$ particle in units of the "magneton" $\mu_M=e\hbar/2M$. A discussion of the general case of arbitrary spin particles will be deferred to a forthcoming publication [8]. In the case of a neutron, we have $g=-3.83$ and $\mu_M=\mu_N=5.05\times10^{-27}$ J/T is the nuclear magneton.

In cylindrical coordinates, the Hamiltonian of the MBS system is given by

$$\begin{aligned}\hat{H}&=-\frac{\hbar^2}{2M}\Delta+V\\ &=-\frac{\hbar^2}{2M}\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}+\frac{1}{\rho^2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}+\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)\\ &\quad+\frac{G}{\rho}\hat{\sigma},\end{aligned} \tag{6}$$

where we introduced the Hermitian operator

$$\hat{\sigma}=\hat{\sigma}_x\sin\phi-\hat{\sigma}_y\cos\phi=\mathrm{i}(\mathrm{e}^{-\mathrm{i}\phi}\hat{s}_+-\mathrm{e}^{\mathrm{i}\phi}\hat{s}_-), \tag{7}$$

the coupling constant ("effective charge")

$$G=\frac{g\mu_M\mu_0 I}{4\pi} \tag{8}$$

and $\hat{s}_\pm=\hat{s}_x\pm\mathrm{i}\hat{s}_y$. If we denote by $\chi_\uparrow$ ($\chi_\downarrow$) the eigenspinors of the $\hat{\sigma}_z$ matrices ($\hat{\sigma}_x\chi_\uparrow=\chi_\uparrow$, $\hat{\sigma}_z\chi_\downarrow=-\chi_\downarrow$), $\hat{s}_\pm$ act as raising and lowering operators:

$$\begin{aligned}&\hat{s}_+\chi_\uparrow=0,\quad \hat{s}_+\chi_\downarrow=\chi_\uparrow,\\ &\hat{s}_-\chi_\uparrow=\chi_\downarrow,\quad \hat{s}_-\chi_\downarrow=0.\end{aligned} \tag{9}$$

The eigenstates of (6), two-component spinors, are denoted by $\Psi(\boldsymbol{r})=\Psi(\rho,\phi,z)$. Because of the translational symmetry of (6) in the z direction, the states Ψ can be factorized according to

$$\begin{aligned}&\Psi_k(\boldsymbol{r})=\frac{1}{\sqrt{L}}\varphi_k(\rho,\phi)\,\mathrm{e}^{\mathrm{i}(2\pi/L)kz},\\ &k=0,\pm1,\pm2,\ldots,\end{aligned} \tag{10}$$

where we assumed periodic boundary conditions for a wire of (macroscopic) length L. Moreover, since $\hat{j}_z=\hat{L}_z+\hat{s}_z$ commutes with $\hat{\sigma}$, the eigenstates (10) can be further classified according to the eigenvalues j_z of $\hat{j}_z$. Given $j_z=m+\frac{1}{2}$, where $m=0,\pm1,\pm2,\ldots$ is the z component of the orbital angular momentum, the most general ansatz for the wavefunction (10) is

$$\begin{aligned}\Psi_k^{(m)}(\boldsymbol{r})&=\frac{1}{\sqrt{4\pi L}}[\varphi_{k\uparrow}^{(m)}(\rho)\mathrm{e}^{\mathrm{i}m\phi}\chi_\uparrow\\ &\quad+\mathrm{i}\varphi_{k\downarrow}^{(m)}(\rho)\mathrm{e}^{\mathrm{i}(m+1)\phi}\chi_\downarrow]\mathrm{e}^{\mathrm{i}(2\pi/L)kz}.\end{aligned} \tag{11}$$

The states (11) are normalized according to

$$\begin{aligned}&\langle\Psi_k^{(m)}(\boldsymbol{r})\,|\,\Psi_{k'}^{(m')}(\boldsymbol{r})\rangle\\ &=\int_0^\infty\rho\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\phi\int_0^\infty\mathrm{d}z\;\Psi_k^{(m)+}(\boldsymbol{r})\,\Psi_{k'}^{(m')}(\boldsymbol{r})\\ &=\delta_{mm'}\delta_{kk'},\end{aligned} \tag{12}$$

provided

$$\int_0^\infty\rho\,\mathrm{d}\rho\,|\varphi_{k\uparrow(\downarrow)}^{(m)}|^2=1. \tag{13}$$

The eigenvalue equation $H\Psi=E\Psi$ leads to the following set of coupled equations,

$$\begin{aligned}&-\frac{\hbar^2}{2M}\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}-\frac{m^2}{\rho^2}-k^2\right)\varphi_{k\uparrow}^{(m)}(\rho)\\ &-\frac{G}{\rho}\varphi_{k\downarrow}^{(m)}(\rho)=E\varphi_{k\uparrow}^{(m)}(\rho),\end{aligned} \tag{14a}$$

$$-\frac{\hbar^2}{2M}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{(m+1)^2}{\rho^2}-k^2\right)\varphi_{k\downarrow}^{(m)}(\rho)$$
$$-\frac{G}{\rho}\varphi_{k\uparrow}^{(m)}(\rho)=E\varphi_{k\downarrow}^{(m)}(\rho). \qquad (14b)$$

To zeroth order in $1/m$ the differential operators in (14) are equal and it is easy to show that in this limit the set of equations (14) admits two distinct classes of solutions denoted by $\varphi_{\pm}^{(m)}(\rho)$: (i) $\varphi_{\uparrow}^{(m)}(\rho)=\varphi_{+}^{(m)}(\rho)$; $\varphi_{\downarrow}^{(m)}(\rho)=-\mathrm{sign}(G)\varphi_{+}^{(m)}(\rho)$ and (ii) $\varphi_{\uparrow}^{(m)}(\rho)=\varphi_{-}^{(m)}(\rho)$; $\varphi_{\downarrow}^{(m)}(\rho)=+\mathrm{sign}(G)\varphi_{-}^{(m)}(\rho)$. As seen from eq. (8) we have: $\mathrm{sign}(G)=\mathrm{sign}(g)\times\mathrm{sign}(I)$. For both choices, (i) as well as (ii), eqs. (14a) and (14b) become degenerate and $\varphi_{\pm}^{(m)}$ satisfies the radial equation of the hydrogen atom in two dimensions [9],

$$-\frac{\hbar^2}{2M}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}\right)\varphi_{\pm}^{(m)}(\rho)$$
$$\pm\frac{|G|}{\rho}\varphi_{\pm}^{(m)}(\rho)=\tilde{E}_{\pm}^{(m)}\varphi_{\pm}^{(m)}(\rho). \qquad (15)$$

Here and in the sequel we suppress the trivial dependence of the wavefunctions and energies on the subscript k. The positive sign in (15) leads to a repulsive potential and does not support any bound states. The choice (ii), however, results in an attractive potential and (15) will exhibit bound states (in the x–y plane) explicitly given by

$$\varphi_n^{(m)}(r)=u_n^{(m)}(\rho)\chi_-^{(m)}(\phi), \qquad (16)$$

where

$$u_n^{(m)}(\rho)=\frac{2\tilde{\gamma}_{nm}}{\sqrt{2n+2|m|+1}}\left(\frac{n!}{(n+2|m|)!}\right)^{1/2}$$
$$\times(2\tilde{\gamma}_{nm}\rho)^{|m|}e^{-\tilde{\gamma}_{nm}\rho}L_n^{(2|m|)}(2\tilde{\gamma}_{nm}\rho) \qquad (17)$$

are the (normalized) states as derived in ref. [9]. Here, the functions $L_n^{(\alpha)}(x)$ are the Laguerre polynomials [10], the spinors $\chi_{\pm}^{(m)}(\phi)$ are defined by

$$\chi_{\pm}^{(m)}(\phi)=\frac{1}{2\sqrt{\pi}}\left[e^{im\phi}\chi_{\uparrow}\mp i\,\mathrm{sign}(G)e^{i(m+1)\phi}\chi_{\downarrow}\right] \qquad (18)$$

and

$$\tilde{\gamma}_{nm}=\frac{M}{\hbar^2}\frac{|G|}{n+|m|+\frac{1}{2}},$$

$$\tilde{E}_{nm}=-\frac{M}{2\hbar^2}\frac{G^2}{(n+|m|+\frac{1}{2})^2}. \qquad (19)$$

At this point we give a physical explanation of why the spinors $\chi_{\mp}^{(m)}$ lead to unbound and bound states respectively. Averaging over the spin variables only, we obtain with the spinors (18)

$$\chi_{\pm}^{(m)+}\boldsymbol{\sigma}\chi_{\pm}^{(m)}=\mp\frac{\mathrm{sign}(G)}{2\pi}\hat{\boldsymbol{e}}_{\phi}, \qquad (20)$$

which means that for the spinors χ_-, corresponding to the bound states, the expectation value of the spin is always aligned with the magnetic field and thus takes maximal advantage of the binding potential. In the state χ_+, on the other hand, spin and magnetic field are always antiparallel, which leads to unbound states.

Motivated by the above discussion of the $m\to\infty$ limit, we will now construct an orthonormal family of trial wavefunctions, $v_n^{(m)}$, which all possess a negative energy expectation value with respect to the Hamiltonian (6). The exact eigenstates of (6), forming a basis of the negative energy space of the Hamiltonian (6), must be at least as numerous as the linearly independent functions $v_n^{(m)}$, thus proving rigorously that (6) possesses at least a doubly infinite countable set of bound states. We write the trial states in the form

$$\Psi_n^{(m)}=v_n^{(m)}(\rho)\chi_-^{(m)}(\phi). \qquad (21)$$

We note that the functions $u_n^{(\lambda)}(\rho)$ obtained by replacing the integer number m by an arbitrary real parameter λ still satisfy the following orthogonality relation:

$$\int_0^\infty \rho\,d\rho\,u_n^{(\lambda)}(\rho)u_{n'}^{(\lambda)}(\rho)=\delta_{nn'}. \qquad (22)$$

This property and the fact that the functions $u_n^{(m)}(\rho)$ are eigenstates of a large part of the Hamiltonian (6) suggest to choose the functions $u_n^{(\lambda)}(\rho)$ as the trial states $v_n^{(m)}(\rho)$ according to

$$v_n^{(m)}(\rho)=u_n^{(\lambda_m)}(\rho), \qquad (23)$$

where we specify λ_m^2 to be the mean value of m^2 and $(m+1)^2$:

$$\lambda_m^2=\tfrac{1}{2}[m^2+(m+1)^2]=m^2+m+\tfrac{1}{2},$$

$$\lambda_m = \sqrt{m^2 + m + \tfrac{1}{4}}. \tag{24}$$

If we use the fact that

$$\left[-\frac{\hbar^2}{2M}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\lambda_m^2}{\rho^2}\right) - \frac{|G|}{\rho}\right] u_n^{(\lambda_m)}(\rho)$$
$$= \tilde{E}_{n\lambda_m} u_n^{(\lambda_m)}(\rho) \tag{25}$$

we find the result

$$\langle \Psi_n^{(\lambda_m)} | \tilde{H} | \Psi_n^{(\lambda_m)} \rangle = \tilde{E}_{n\lambda_m} < 0, \tag{26}$$

where $\tilde{H}$ is the Hamiltonian $\hat{H}$ (see eq. (6)) apart from the kinetic energy in the z direction,

$$\tilde{H} := \hat{H} - \frac{\hbar^2}{2M}\frac{\partial^2}{\partial_z^2}. \tag{27}$$

We thus proved that for each given value of the magnetic quantum number m there is an infinity of mutually orthogonal variational states which all belong to negative values of the mean energy (in x–y direction), i.e., they are approximations of bound quantum mechanical states.

The exact solutions of the coupled system of equations (14) can be constructed by solving (14a) for $\varphi_{\downarrow}^{(m)}(\rho)$, and inserting the result in (14b). The spectrum of (6) is obtained by solving the resulting fourth order differential equation together with the boundary condition of normalizability of its solutions. The calculations are complicated and will be presented in a later publication [8]. Here it suffices to say that the exact binding energies of the Hamiltonian (6) are not much different from the variational estimates (26), and approach (26) in the limit $|m| \to \infty$.

The energies in (26) can be considered as variational estimates of the exact binding energies of the MBS system. An estimate for the extension of the wavefunctions in ρ direction is given by

$$l_{nm} = 1/\gamma_{nm} \approx 1/\tilde{\gamma}_{n\lambda_m}. \tag{28}$$

In the case of a neutron ("N") we have

$$E_{nm}^{(\mathrm{N})}/h \approx \tilde{E}_{n\lambda_m}^{(\mathrm{N})}/h = -\frac{M_\mathrm{N} G_\mathrm{N}^2}{2h\hbar^2}\frac{1}{(n+\lambda_m+\frac{1}{2})^2}$$
$$\approx -420\ \mathrm{MHz}\frac{I[\mathrm{A}]^2}{(n+\lambda_m+\frac{1}{2})^2},$$
$$l_{nm}^{(\mathrm{N})} \approx l_{n\lambda_m}^{(\mathrm{N})} \approx 35\ \text{Å}\,\frac{n+\lambda_m+\frac{1}{2}}{I[\mathrm{A}]}, \tag{29}$$

where the current I is given in amperes. These estimates show that (a) the transition energies are in the rf regime and (b) that for reasonable currents in the wire (≈ 1 mA) the extension of the wavefunction in ρ direction is in the μm regime. This means that very thin wires are required. An alternative way of producing the magnetic field would be to use a straight, well collimated electron beam which would be obligatory in the case that neutron absorption cannot be neglected in the wire. Such narrow electron beams should be well within the experimental reach since mesoscopic structures on a scale of some 100 nm can be manufactured by electron beam lithography [11]. One could also think of a crystallized electron beam in a circular trap [12], which would yield an electron beam with the smallest possible diameter and a minimum of fluctuations.

Due to the small excitation energies it might take a very long time to populate the MBS ground state by spontaneous radiative decay. The particles can, however, very effectively be trapped in the various bound states of the wire, by immersing the wire into a cold cloud of neutrons and suddenly switching on the current I in the wire. It is tempting to conjecture that complete trapping of the particles might be achieved by closing the wire to a loop. This point is currently investigated in more detail [8].

Apart from merely demonstrating the curious fact that a neutral particle can form an infinity of magnetically bound states, several applications of the MBS system might be envisaged:

(i) Determination of the particles' magnetic moment by studying (induced) transitions between bound states. In the case of stable particles this method can be considered as an alternative method to existing ones. In the case of unstable particles it might provide an experimental means of measuring these magnetic moments. Obviously, the method is restricted to those particles whose life time is larger than the typical inverse transition time between levels.

(ii) Making use of the free motion in z direction, the MBS set-up could be used as a quantum mechanical neutron wave guide.

(iii) In the case of neutrons, and for reasonable currents in the wire, the structure of the MBS system resembles very closely a hydrogen Rydberg atom and might be used in the study of quantum chaos [13,14].

The MBS system is simpler since the neutrons can be excited by changing (periodically) the current in the wire. This is equivalent to changing the electric charge in the hydrogen problems, a new mode of excitation obviously not accessible to conventional quantum chaos experiments with Rydberg atoms [13,14].

(iv) Since neutrons are fermions, not more than one fits into a given state with specified quantum numbers (n, m, k). Analogous to the SSE system, the z degree of freedom is necessary to get a macroscopic population of an (n, m) state and a measurable signal for absorption spectroscopy. As already mentioned in the introduction, many body effects of purely spin correlated systems can be studied with the MBS set-up. If the neutrons are replaced by spin-1 particles, even Bose condensation could be studied. In an optimistic view of the problem one may also expect that small effects like the gravitational interaction may have a measurable effect on the quantum mechanical energy levels, as is the case for Paul's magnetic bottle whose construction and working principles are described in ref. [15] and applications to the gravitational effect in ref. [16].

On the other hand, there are problems which are of great importance for the practicability of the MBS system: The absorption of neutrons in the wire, the influence of inhomogeneities of the wire on the magnetic field, or, in the case that an electron beam is used to replace the current in the wire, the inhomogeneities of the electron beam resulting in fluctuations of the current I whose spectral power could be enough to induce transitions between the bound states. We shall treat these aspects in more detail in a later paper [8].

In conclusion we feel that the MBS system has an interesting interdisciplinary character in that it involves elements from nuclear, atomic and solid state physics and might also provide another model for studying quantum chaos.

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