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# On the integrability of the two-ion Paul trap in the pseudo potential approximation

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The two-ion Paul trap is integrable in the pseudo potential approximation for trap asymmetry parameters $\lambda=\frac{1}{2}$ and $\lambda=2$. Integrability at $\lambda=\frac{1}{2}$ (proved by explicitly stating the corresponding analytical integral of the motion) is not consistent with recently published results [G. Baumann, Phys. Lett. A 162 (1992) 464].

In a recent publication [1] Baumann suggests that the two-ion Paul trap is chaotic in the pseudo potential approximation for the trap asymmetry parameter $\lambda=\frac{1}{2}$ and nonzero angular momentum parameter $\nu$. Moreover, a positive Lyapunov exponent was calculated numerically for this case [1]. Both results are surprising since it is possible to construct analytically an explicit integral of the motion.

The problem is the following: Given is the Hamiltonian

$$H=\tfrac{1}{2}p_\rho^2+\tfrac{1}{2}p_\zeta^2+V(\rho,\zeta)\,, \tag{1}$$

with

$$V(\rho,\zeta)=\tfrac{1}{2}\rho^2+\tfrac{1}{2}\lambda^2\zeta^2+\frac{\nu^2}{2\rho^2}+\frac{1}{(\rho^2+\zeta^2)^{1/2}} \tag{2}$$

and $p_\rho=\dot\rho$, $p_\zeta=\dot\zeta$. This Hamiltonian describes the relative motion of two charged particles in a Paul trap in the pseudo potential approximation with $\lambda$ and $\nu$ related to the asymmetry of the time averaged trapping potential and the relative angular momentum, respectively [2]. It can be interpreted as the Hamiltonian of a single particle moving in two dimensions, $\rho$ and $\zeta$, respectively.

The equations of motion derived from (1) are

$$\ddot\rho=\frac{\nu^2}{\rho^3}-\rho+\frac{\rho}{(\rho^2+\zeta^2)^{3/2}}\,, \tag{3}$$

$$\ddot\zeta=-\lambda^2\zeta+\frac{\zeta}{(\rho^2+\zeta^2)^{3/2}}\,. \tag{4}$$

The Hamiltonian $H$ is conservative and autonomous. Therefore, $E$, the total energy of the system, is a constant of the motion [3]. Besides the energy and for arbitrary $\nu$ two further integrals of the motion exist for $\lambda=2$ and $\lambda=\frac{1}{2}$, respectively. This fact was first stated in ref. [2]. Unfortunately, the corresponding integrals, $F$ and $G$, respectively, were not reported correctly in ref. [2]. The integral $F$, which applies for the case $\lambda=2$, is given by

$$F(\rho\dot\rho,\zeta\dot\zeta;\nu)=\zeta\dot\rho^2-\dot\zeta\rho\dot\rho+\frac{\zeta}{(\rho^2+\zeta^2)^{1/2}}-\rho^2\zeta+\frac{\nu^2\zeta}{\rho^2}\,. \tag{5}$$

This integral was correctly reproduced and stated in ref. [1]. Also, the connection to Noether's theorem was pointed out [1].

For $\lambda=\frac{1}{2}$ the integral of the motion is given by

$$G(\rho\dot\rho,\zeta\dot\zeta;\nu)=I_\rho^2+I_\phi^2+\nu^2(\rho^2+\zeta^2)\,, \tag{6}$$

where

$$I_\rho=\frac{\nu^2}{\rho}+\rho\dot\zeta^2-\dot\rho\zeta\dot\zeta+\frac{\rho}{(\rho^2+\zeta^2)^{1/2}}-\tfrac{1}{4}\zeta^2\rho \tag{7}$$

and

$$I_\phi=-\frac{\nu}{\rho}(\rho\dot\rho+\zeta\dot\zeta)\,. \tag{8}$$

With the help of (3) and (4) it can be proved easily by direct differentiation that indeed $dG/dt=0$. This means that for $\lambda=\frac{1}{2}$ and for arbitrary $\nu$ two integrals of the motion exist ($F$ and $G$). Therefore, $H$ is integrable for $\lambda=\frac{1}{2}$. Integrability of $H$ is equivalent with absence of chaos [3]. On the basis of the above result the chaos and the positive Lyapunov exponent reported in ref. [1] in the case $\lambda=\frac{1}{2}$, $\nu\neq 0$ cannot be understood.

The remarkable feature about the Hamiltonian $H$ is that the constants $F$ and $G$ are not related to trivial symmetries of the problem. The constants are due to the existence of nontrivial generalized symmetries in the sense of Noether [1]. Nontrivially integrable Hamiltonians, like the Toda lattice [3], are extremely rare in the physics literature. Therefore, the Hamiltonian $H$ defined in (1) may deserve some further attention.

## References

[1] G. Baumann, Phys. Lett. A 162 (1992) 464.
[2] R. Blümel, C. Kappler, W. Quint and H. Walther, Phys. Rev. A 40 (1989) 808.
[3] A.J. Lichtenberg and M.A. Lieberman, Applied mathematical sciences, Vol. 38. Regular and stochastic motion (Springer, Berlin, 1983).