Blümel Replies: I do not hesitate to admit that my recent Letter [1] stirred critical comments by many scientists. While no formal errors were pointed out, many scientists are unhappy with the interpretation of the Stern-Gerlach (SG) experiment proposed in [1]. The two main points are summarized in Schack's Comment [2]: (i) absence of exponential sensitivity in the quantum states and (ii) the use of a logarithmic time scale.

The main point of [1] is to study a sequence of quantum propagators that follow the recursion relation

$$\hat{U}_{n+1} = \hat{U}_n \hat{U}_{n-1} \tag{1}$$

It is shown that there are cases where (1) is equivalent with a chaotic mapping Q that is reminiscent of the cat map. Consequently, chaos and exponential sensitivity can occur in the matrix elements of (1) with respect to system parameters as a function of the discrete index n. It is important to note here that this fact is not disputed. It is important to note here that this fact is not disputed. The problems raised by Schack and other scientists do not concern this fact but the legitimate question of whether, for instance, because of (i) and (ii) above, the chaos induced by (1) is interesting at all.

Schack's observation (i) is correct and undisputed. I would like to point out that an essentially identical discussion appears in my paper [3] cited in [1] as "to be published." Mindful of the absence of sensitivity with respect to the initial conditions in the quantum states [3] the whole point of [1] is to show that sensitivity and chaos can nevertheless occur in the matrix elements of quantum operators. I do not think that the sentence on p. 430 of [1] quoted by Schack may be interpreted as a possible source of confusion since the following two sentences on p. 430 of [1] make it very clear that the angles β refer to the magnetic field strengths and therefore to system parameters.

Let us now turn to the more serious point (ii). Here we have to distinguish clearly between the properties of the recurrence (1) and its physical implementations as experimentally accessible systems. While there is no doubt that (1) exhibits chaos and sensitivity, problems arise when we consider the implementations.

I agree with Schack that the positions φ_n of a free particle rotating on a ring, clocked at exponentially separated times t_n, behave in essentially the same way as the angles β_n. But as *systems* the ring and the SG apparatus are not identical. With the ring, we introduce an additional dynamical variable, the rotating particle's angular momentum L. The particle's phase space is then two dimensional and although φ_n is a chaotic sequence the ring *system* is not chaotic since as a classical Hamiltonian system it is integrable which means here that the points (φ_n, L) lie on one-dimensional straight lines in the two-dimensional phase space [4]. In the SG example, however, the angle β is the only variable. Therefore, I feel entitled to call the SG system chaotic in n.

Spacing the magnets exponentially close in n, the flight time t is proportional to n and the SG system is also chaotic in t. There are obvious physical limitations for this procedure. But since $n \approx 10$ is enough to see the predicted sensitivity it can be implemented.

In [1] I discuss a more elegant implementation of (1) as a spin-1/2 system quasiperiodically perturbed by a sequence of field pulses. It was shown in [5] that the Fourier transform of a train of pulses constructed according to (1) contains a set of *countably* infinite frequencies. Moreover, the frequency content in a drive satisfying (1) can be made effectively finite by appropriately "engineering" the drive [5]. This observation puts the SG system into the context of quasiperiodically driven systems already under active experimental investigation [6].

Finally, I would like to mention that Fox [7] has pointed out that the exponential sensitivity in the SG system discussed in [1] is indeed experimentally measurable if two conditions are met. (a) The orientation of the spin has to be measured after every block of magnets n, and (b) the initial angles β_0 and β_1 have to be very small. With $\beta_0 = \beta_1 = 1$, as chosen in [1], the variance of the quantum spin saturated too early, effectively masking the exponential sensitivity in n.

R. Blümel
Laboratory for Plasma Research
University of Maryland
College Park, Maryland 20742

Received 23 March 1995
PACS numbers: 05.45.+b

[1] R. Blümel, Phys. Rev. Lett. **73**, 428 (1994).
[2] R. Schack, preceding Comment, Phys. Rev. Lett. **75**, 581 (1995).
[3] R. Blümel, Nucl. Phys. **A572**, 9 (1994).
[4] The importance of distinguishing between formally chaotic sequences and the properties of systems was pointed out to me by T. Geisel and R. Ketzmerick.
[5] J.M. Luck, H. Orland, and U. Smilansky, J. Stat. Phys. **53**, 551 (1988).
[6] A. Haffmans, R. Blümel, P.M. Koch, and L. Sirko, Phys. Rev. Lett. **73**, 248 (1994).
[7] R.F. Fox (private communication).