Experimental Identification of Non-Newtonian Orbits Produced by Ray Splitting in a Dielectric-Loaded Microwave Cavity

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We use a thin dielectric-loaded microwave cavity to identify experimentally the signature of non-Newtonian periodic orbits caused by ray splitting at sharp, dielectric interfaces. In our experiments non-Newtonian orbits manifest themselves as distinctive peaks in the Fourier transform of the frequency spectrum, which cannot be assigned to any Newtonian periodic orbits. Because the electromagnetic Helmholtz and quantal Schrödinger equations are equivalent in two dimensions, our results are directly relevant to quantum chaos studies. [S0031-9007(97)02910-4]

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The behavior of the electromagnetic field at an interface between dielectric media is of fundamental importance in optics. In the geometric optics limit reflection and transmission at sharp dielectric interfaces give rise to the phenomenon of ray splitting (RS) currently discussed in the context of acoustics and quantum chaos [1–3]. It has been shown theoretically in the context of quantum mechanics [3] that ray splitting is accompanied by the occurrence of non-Newtonian trajectories whose signatures are clearly present in the Fourier spectrum of the oscillating part of the level density of the system. Non-Newtonian orbits also appear in the context of diffraction [4], a topic closely related to the RS phenomena discussed in this paper.

The central point of this Letter is the experimental demonstration of the existence of non-Newtonian RS orbits in the microwave context. Because the electromagnetic Helmholtz and quantal Schrödinger equations are equivalent in two dimensions, our results are directly relevant to quantum chaos studies [2].

In quantum mechanics, ray splitting occurs whenever the potential changes on a scale much shorter than the local de Broglie wavelength. The key features of RS phenomena are conveniently illustrated via the following one-dimensional quantal scattering problem. A particle P of mass m and energy $E > U_0$ incident from the left scatters off the potential $U(x) = U_0/[1 + \exp(-x/w)]$ [5]. U_0 is the strength of the potential and w is the width of the potential step. Asymptotically the wave function is given by $\psi(x) = \exp(ikx) + r\exp(-ikx)$ for $x \to -\infty$, and $\psi(x) = t\exp(i\kappa x)$ for $x \to \infty$, where $k = [2mE]^{1/2}/\hbar$ and $\kappa = [2m(E - U_0)]^{1/2}/\hbar$. The reflection coefficient r, computed explicitly in Ref. [5], is

$$r = \frac{\sinh[\pi w(k - \kappa)]}{\sinh[\pi w(k + \kappa)]}. \qquad (1)$$

We are interested in the quasiclassical limit that involves the scattering of waves for $\hbar \to 0$ in the limit of a sharp potential step, i.e., $w \to 0$. The resulting double limit $\hbar \to 0, w \to 0$ is undefined without further specification. There are two cases of interest: (i) For fixed w we let $\hbar \to 0$ and only then let $w \to 0$. This is the "Newtonian limit." In this limit the ray dynamics of P is ordinary Newtonian mechanics; for $E > U_0$ ($E < U_0$), P is transmitted (reflected) with unit probability to the right (left). (ii) For fixed $\hbar$ we let $w \to 0$ and only then let $\hbar \to 0$. This is the RS limit. In this case the ray dynamics of P is a non-Newtonian, nondeterministic mechanics that involves ray splitting at the potential step [1]. The RS limit of (1) is $r = (k - \kappa)/(k + \kappa)$, which, it should be noted, is independent of $\hbar$. Consequently, even for $E > U_0$ and $\hbar \to 0$, P is not transmitted with unit probability; it is reflected with finite probability to the left. It was shown in [1,3] that non-Newtonian trajectories are associated with this reflection phenomenon. Furthermore, it was shown that the corresponding non-Newtonian trajectories are quite "real" in the sense that they manifest themselves as marked peaks in the Fourier transform of the level density of RS quantum systems and should contribute appreciably to Gutzwiller's trace formula [6]. It was, therefore, suggested in Ref. [1] to modify the trace formula for RS systems by including the contribution from split rays according to

$$\tilde{\rho}(E) = \mathrm{Im} \sum_n \frac{A_n^{1/2} T_n}{2i\hbar \sinh(\lambda_n/2)} \exp\left[i\left(\frac{S_n(E)}{\hbar} + \phi_n \right) \right]. \qquad (2)$$

In this formula $\tilde{\rho}$ is the oscillating part of the level density, the sum extends over all periodic orbits of the system (Newtonian and non-Newtonian) including repetitions, T_n are the traversal times of the primitive periodic orbits, λ_n are the associated stability indices, $S_n(E)$ are the classical actions at energy E, ϕ_n are phases, and

$$A_n = \left[\prod_{i=1}^{\varrho_n} |r_i|^2 \right] \left[\prod_{j=1}^{\tau_n} (1 - |r_j|^2) \right], \qquad (3)$$

where ϱ_n counts the number of reflections, τ_n counts the number of transmissions of orbit number n, and $|r_i|^2$ is the reflection probability at encounter number i at an RS interface. The suggested [1] modification (2) of the Gutzwiller formula differs from the standard Gutzwiller formula [6] by the weights (3). We see that for reflection probabilities all small, the weights (3) of non-Newtonian RS orbits are essentially proportional to the product of the $|r_i|^2$. Therefore, for a particular non-Newtonian orbit to be important, the product of the reflection probabilities should not be too small. This translates physically into the requirement that the width of the RS interface has to be very small compared to the local wavelength. At $E = 2U_0$, e.g., the reflection probability (1) reaches the one-percent level only if the width of the potential w is smaller than about $1/20$ of a wavelength. This requirement is easily fulfilled in the microwave context where typical wavelengths are in the cm regime and the widths of dielectric interfaces can be machined to μm precision. While for a Newtonian orbit reflection and transmission are decided on the basis of energy and momentum considerations only, reflection and transmission in the RS limit are stochastic processes governed by the reflection probabilities $|r_i|^2$ that characterize the nondeterministic element in the ray dynamics of RS problems. If the suggestion (2) offered in Ref. [1] is true, a Fourier transform of $\tilde{\rho}(E)$ should show peaks not only at the actions of the Newtonian periodic orbits but also at the actions of non-Newtonian RS orbits with weights that depend on the reflectivity of the potential. This is the guiding principle we use in this Letter to demonstrate the signature of non-Newtonian orbits in a thin microwave cavity containing a bar of Teflon. Similar experiments were suggested in Ref. [2] for the investigation of RS effects.

It is well known [7] that in the case of thin cavities the Maxwell equations reduce to a Helmholtz equation in two dimensions xy. An empty cavity is "thin" for frequencies ν less than the cutoff frequency $\nu_c = 2c/L$, where c is the speed of light and L is the cavity height in the z direction. Thin microwave cavities are excellent models for quantum chaos [8–11]. The new point in our experiment is to introduce a bar of dielectric (Teflon) in order to generate non-Newtonian RS orbits. The dielectric constant of Teflon is known experimentally [12] to be $\epsilon = 2.08$, with essentially no frequency dependence over the range of interest in this Letter [13]. The exact shape of our cavity (a Bunimovich stadium [6]) and of the inserted dielectric bar is shown in Fig. 1, for one of the placements we used. In a first series of experiments we used a transmission method [14] to measure the resonance frequencies of the empty cavity (no dielectric bar). We were careful to compare spectra obtained with different placements and insertion depths of the coupling antennas to ensure that we missed no levels up to a certain frequency. We obtained 50 resonance frequencies $\nu_j, j = 1, \ldots, 50$. The corresponding wave numbers are $k_j = 2\pi\nu_j/c$. We define the "energies" $E_j = k_j^2$. In order to extract the oscillating part of the level density $\tilde{\rho}$ from the measured data, we need the mean $\overline{N}(E)$ of the staircase function, i.e., the number of resonances up to the energy E. Theoretically, the first two terms in a systematic expansion of $\overline{N}(E)$ in powers of E are given by [6] $\overline{N}(E) = \alpha E - \beta\sqrt{E}$, where $\alpha = A/4\pi$, $\beta = P/4\pi$, A is the area of the stadium, and P is its perimeter. With the measured dimensions of the cavity given in Fig. 1 and a conservative estimate of their uncertainties, we obtain $\alpha = 0.00220(5)$ and $\beta = 0.0487(8)$. A least squares fit of the measured staircase of cavity resonances yields $\alpha = 0.00222$ and $\beta = 0.0489$; both agree with theoretical expectations based on measured dimensions. In the case of billiards the classical actions in (2) are given by $S_n(E) = \oint_{\gamma_n} \vec{p} \cdot d\vec{x} = kl_n$, where the l_n are the lengths of the periodic orbits γ_n. Since the actions scale as $\sqrt{E}$, and the round-trip times $T_n(E)$ scale as $1/\sqrt{E}$, a Fourier transform of $\tilde{\rho}(E)$ shows peaks at the lengths l_n of periodic orbits. We determined the density of states for the empty Bunimovich stadium according to $\rho(E) = \sum_{j=1}^{50} \delta(E - E_j)$, subtracted the mean density $\overline{\rho}(E) = d\overline{N}(E)/dE$ to obtain $\tilde{\rho}(E)$, and computed

$$\mathcal{F}(l) = \int_0^{E_{\max}} \tilde{\rho}(E)\omega(E)\exp(-ikl)\, dE\,, \qquad (4)$$

where $E_{\max} = E_{50}$ and $\omega(E) = \sin(E/E_{\max})$ is a window function that suppresses the Gibbs overshoot phenomenon [15]. The absolute square of $\mathcal{F}(l)$ is shown in Fig. 2. We see pronounced peaks that, as expected, occur near the lengths of certain periodic orbits. As shown in Fig. 2 the large peak at $l \simeq 52$ cm covers several unresolved periodic orbits; we estimate that more than 1000 levels would be needed to resolve them in this cluster. It is important to note that on the scale of Fig. 2 no peak occurs below $l = 20$ cm.

We turn now to the case with the dielectric inserted into the cavity. We studied two cases: (D_1) $d = 9.55$ cm; (D_2) $d = 7.17$ cm. For each case we measured the first 50 resonance frequencies and performed the windowed

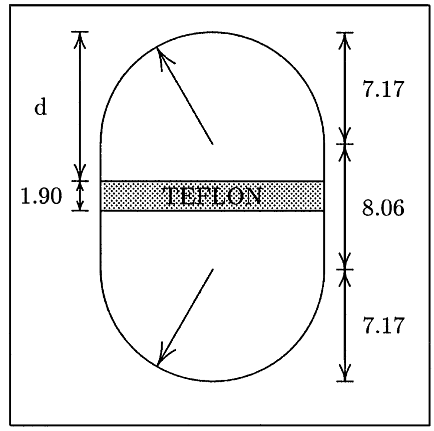

FIG. 1. Sketch of the microwave cavity in the xy plane. Dimensions are given in cm. To study the shift in position of periodic non-Newtonian RS orbits, the position d of the Teflon dielectric bar, which entirely fills the 0.8 cm height of the cavity, can be shifted along the straight section of the cavity. The position shown ($d = 9.55$ cm) corresponds to the case D_1 described in the text.

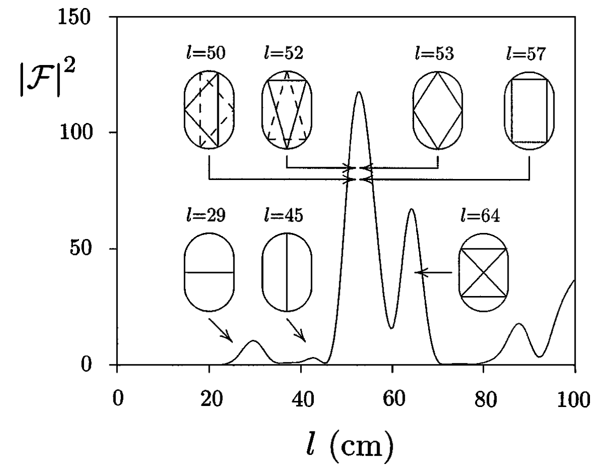

FIG. 2. Absolute square of the Fourier transform of the fluctuating part of the density of resonances of the empty cavity shown in Fig. 1. The assignment of peaks of $|\mathcal{F}(l)|^2$ to simple, periodic orbits is shown along with the length l of each such orbit in cm.

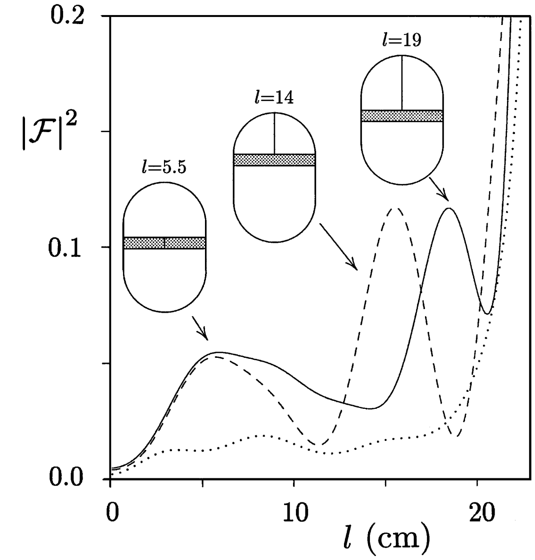

FIG. 3. Absolute square of the Fourier transform of the fluctuating part of the density of resonances on an expanded vertical scale over the interval $0 < l < 23$ cm. Dotted line: Empty cavity. Solid line: Teflon dielectric inserted (case D_1, $d = 9.55$ cm). Dashed line: Teflon dielectric inserted (case D_2, $d = 7.17$ cm). The assignment of peaks in the transform to simple, periodic, non-Newtonian RS orbits is shown along with the optical path length l for each such orbit given in cm.

Fourier transform as described above. It is important to realize that the Teflon bar exerts a *global* effect on the frequency spectrum: It shifts the position of *all* resonances in the frequency domain. For the discussion below, one should not try to focus on what adding the Teflon bar might do to any individual resonance in the frequency domain; rather, one should focus on what it does to the Fourier transform of the global frequency spectrum. While the appearance of the transform for $l > 20$ cm is essentially unchanged, striking new features appear for $l < 20$ cm. On an expanded scale, Fig. 3 shows $|\mathcal{F}(l)|^2$ for the empty cavity (dotted line) and the two dielectric cases, (D_1) (full line) and (D_2) (dashed line). While even on the expanded scale the transform for the empty stadium shows no significant structure below $l = 20$ cm, the case D_1 shows two peaks, one at $l \simeq 5.5$ cm and another at $l \simeq 19$ cm. The peak at $l \simeq 5.5$ cm can be explained as the signature of a family of periodic non-Newtonian orbits bouncing inside the dielectric bar parallel to the major axis of the stadium. Inside the dielectric the action of a periodic orbit is multiplied by the index of refraction. In other words, the length of a periodic orbit is not the geometric length but the optical path length. With the dimensions given in Fig. 1 we predict that the optical path length of this orbit bouncing inside the Teflon bar is $l_{\text{opt}} = 2 \times \sqrt{\epsilon} \times 1.9$ cm $= 5.5$ cm. This is in excellent agreement with the location of the first peak of D_1. The second peak of D_1 can be interpreted as a non-Newtonian RS orbit bouncing between the round tip of the cavity and the nearest edge of the Teflon bar [16]. Since this orbit travels entirely outside the dielectric, its predicted optical path length is equal to its geometric length $l = 2 \times 9.55$ cm $= 19.1$ cm. This agrees well with the experimental data. Note that shifting the location of the Teflon bar should not influence the location of the non-Newtonian “internal bounce” orbit at $l \simeq 5.5$ cm. Indeed, the first peak in the Fourier transform of D_2 occurs at the same position as the peak in the D_1 data. The non-Newtonian orbit bouncing outside the dielectric, however, is expected to shift to the new position $l = 2 \times 7.17$ cm $= 14.34$ cm. Experimentally, the second peak in D_2 does occur near this expected position.

The reason for the non-Newtonian nature of the internal and external bounce orbits identified in our experiments is the following. As shown in Ref. [2], the Helmholtz equation for the 2D cavity (partially) filled with dielectric can be interpreted as a Schrödinger equation with an attractive potential over the extension of the dielectric. Since an internal bounce orbit is one that bounces at normal incidence and at *positive* energy *inside* an everywhere negative (attractive) potential, it cannot correspond to a Newtonian orbit. The same reasoning applies to orbits bouncing off the dielectric interface at normal incidence from the outside. In the Newtonian case a trajectory normally incident on an attractive potential is transmitted with probability one. Therefore, the peaks observed near $l = 19$ cm and $l = 14$ cm cannot originate from Newtonian periodic orbits.

Locations, widths, and heights of the non-Newtonian peaks shown in Fig. 3 can be estimated semiclassically. In order to illustrate the procedure, we focus on the non-Newtonian external bounce orbit, which, for case D_2, occurs at $l \approx 14$ cm. On the basis of the theoretical framework provided in Refs. [2,3], we obtain its contribution to the oscillating part of the scaled density of resonances as $\tilde{\rho}_s(E) = (Rr/2\pi k)\sin(2kR)$, where $r = (1 - \sqrt{\epsilon})/(1 + \sqrt{\epsilon})$ is the reflection coefficient and $R = 7.17$ cm is the radius of the circular endcap of the cavity (see Fig. 1). The Fourier transform (4) of $\tilde{\rho}_s(E)$ can be performed analytically for $\omega = 1$. Neglecting an additive nonresonant term, we obtain

$$|\mathcal{F}(l)|^2 = \frac{R^2 r^2}{4\pi^2} \frac{\sin^2[(2R - l)K/2]}{[(2R - l)/2]^2}, \qquad (5)$$

where $K = \sqrt{E_{50}} \approx 1.54$ /cm. For large K (5) becomes

$$|\mathcal{F}(l)|^2 \to \frac{R^2 r^2}{2\pi} K\delta(2R - l). \qquad (6)$$

This result shows that in the limit of large K, isolated non-Newtonian orbits indeed contribute a δ-function singularity to $\mathcal{F}(l)$ as conjectured above on the basis of (2). In addition (6) confirms that the location of the external bounce orbit is indeed expected at $l = 2R \approx 14$ cm. For finite K we obtain broadened peaks in l with full width at half maximum (FWHM) of $\Gamma \approx 5.57/K$. In the present case $\Gamma \approx 3.6$ cm, which agrees well with the widths of the peaks at $l = 14$ cm and $l = 19$ cm in Fig. 3. The strength of a peak is defined as $s = \int |\mathcal{F}(l)|^2 \, dl$. According to (6) we should expect $s = 0.41$ cm. Estimating the strength of the peak at $l = 14$ cm in Fig. 3 as its height times its FWHM, we obtain $s \approx 0.43$ cm. The experimental result and the theoretical estimate for the strength are close. Though encouraging, this agreement is somewhat fortuitous for the following reasons. (i) The Fourier transform of the experimental resonances contains a window function. If it is taken into account, it reduces the theoretical estimate of the strength by a factor of 4. (ii) Over the whole range of the 50 resonances used in the Fourier transform, the level density $\tilde{\rho}_s(E)$ exhibits only about three oscillations. We expect that the full information on the non-Newtonian orbit is not yet contained in such a small number of oscillations. Consequently, the heights of the peaks in Fig. 3 are not yet converged in the number of resonances. We confirmed this by computing $|\mathcal{F}(l)|^2$ with inclusion of 40, 45, and 50 resonances in the transform (4).

On the basis of the above discussion we conclude that although our experiment could not measure the converged heights of the non-Newtonian peaks in Fig. 3, it did establish beyond reasonable doubt their existence, widths, and approximate locations.

In this Letter, we presented first experimental evidence for the existence of non-Newtonian orbits produced by ray splitting in a dielectric-loaded microwave cavity. Since ray splitting occurs in all wave systems with sharp, partially transparent interfaces, we expect the appearance of non-Newtonian RS orbits as a universal feature of all RS wave systems.

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[13] The loss of Teflon is so low (loss tangent $\lesssim$0.0004; see Ref. [12]) that we found that introducing the Teflon bar into the present cavity did not lower the average $Q = \nu/\Delta\nu$ of the observed resonances. In fact, we found that the average Q rose slightly because the Teflon increased the effective volume of the cavity.
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[15] See J. S. Walker, *Fast Fourier Transforms* (CRC Press, Boca Raton, 1991). We checked that all non-Newtonian orbits discussed in this Letter were insensitive to the choice of window function $\omega(E)$. We reproduced essentially the same locations and relative heights of the peaks shown in Fig. 3 with $\omega(E) = \sin(\sqrt{E/E_{\max}})$, $\sin^2(\sqrt{E/E_{\max}})$, and $\sin^2(E/E_{\max})$.
[16] There will also be other peaks, e.g., caused by a non-Newtonian RS orbit bouncing between the round tip of the cavity and the farthest edge of the Teflon bar. Its length for case D_1 (D_2) would be 24.6 cm (19.8 cm), causing it to be "lost" in the giant peak rising near $l = 20$ cm in Fig. 3. The peaks discussed in the text associated with the shortest non-Newtonian RS orbits are the only ones easily resolved in the present geometry.